

Mathematical and Statistical Modeling of Uncertainty: A Comprehensive Review of Theoretical Foundations and Applied Methods

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ABSTRACT:

Uncertainty affects scientific research, engineering design, economic forecasting, healthcare decision-making, climate modeling, artificial intelligence, and policy analysis, but its mathematical and statistical treatment remains fragmented across disciplines. This review provides a comprehensive, non-systematic synthesis of theoretical foundations and applied methods for modeling, quantifying, propagating, and interpreting uncertainty. Foundational books, seminal studies, and recent peer-reviewed literature were reviewed conceptually and thematically. The article examines uncertainty classifications, mathematical frameworks, statistical techniques, propagation methods, complex-system modeling, applied domains, comparative evaluation, challenges, and future directions. The review shows that uncertainty arises from randomness, incomplete knowledge, model assumptions, parameter estimation, measurement error, and computational limitations. Probability theory, fuzzy sets, interval analysis, evidence theory, and possibility theory provide complementary representations, while Bayesian inference, Monte Carlo simulation, bootstrap resampling, and sensitivity analysis support quantification and propagation. Applications extend across engineering reliability, finance, climate modeling, healthcare, artificial intelligence, and supply chain systems. Future research should emphasise hybrid, interpretable, scalable, and application-sensitive frameworks that integrate probabilistic, non-probabilistic, computational, and data-driven approaches for reliable decision-making under complex uncertainty.

Keywords: Uncertainty modeling, Uncertainty quantification, Probability theory, Bayesian inference, Monte Carlo simulation, Machine learning uncertainty

1. Introduction

Uncertainty is an integral component of scientific research, engineering design, economic prediction, health care decision-making, climate modelling, AI and policy analysis. In real-world systems, the

information available for decision making can be incomplete, observations can be noisy, models may be imperfect, and environments can change. Mathematical and statistical models offer the

means to express, quantify, propagate and reduce uncertainty in such situations. (Morgan et al., 1990). As modern systems are getting more and more complex, data-driven and computation-intensive, the value of uncertainty modeling is getting larger. (R. Ghanem et al., 2017). It is important to have a clear separation of similar concepts. Often risk is used to describe those scenarios where the possible outcome and the likelihood of each are known, while uncertainty is used to describe the case where either the outcome, the probability or the behaviour of the system is not known. (Knight, 1921). Variability is related to natural randomness, natural heterogeneity and ignorance is related to lack of knowledge, which may be diminished with improvements in data, models or expert judgment. Ambiguity is the state of being unclear or the state of conflicting information regarding concepts, evidence or interpretations. These differences are significant because the appropriate mathematical and statistical techniques to deal with the different types of uncertainty vary. (Walker et al., 2003).

Although there has been significant research and development, uncertainty modelling is still fragmented and exists in many disciplines. The engineering field and the fields of climate science, economics, healthcare, artificial intelligence and decision sciences have different terminologies, assumptions and methodological traditions. In risk analysis, for instance, probabilistic models are used, while in the context of linguistic uncertainty fuzzy methods are used, in the context of bounded uncertainty interval, interval methods are used, and in the context of evidence-based updating, Bayesian approaches are used. This diversity is beneficial but also brings about a division in concepts and methods. Another issue is that there is no commonality of view on how mathematics and statistics can complement each other. These uncertainties are often considered independently, but in interacting ways in complex systems (Oberkampf et al., 2002): model uncertainty, parameter uncertainty, measurement uncertainty, and numerical uncertainty. Theoretically, aleatory and epistemic uncertainty can be separated, but in practice, it is not always

easy to distinguish between them (Der Kiureghian & Ditlevsen, 2009).

Although there is good literature on the individual traditions of uncertainty modelling, there is a lack of integrated modelling in the fields of probability, fuzzy, interval, evidence, Bayesian, Monte Carlo simulation, sensitivity analysis and machine learning-based uncertainty modelling. Most research is either theoretical in nature or it is more quantitative in terms of applications, but not linking conceptual foundations with computational and domain-specific ones. New developments in deep learning, scientific machine learning and engineering prognostics have further extended the field, and there is a need for a single point of reference to review classical and modern approaches. (Abdar et al., 2021). Another important aspect is the need to bridge the gap between the uncertainty-aware artificial intelligence and physics-informed methods and existing uncertainty frameworks in the field of statistics and mathematics. (Psaros et al., 2023). The review provides a synthesis of the key concepts, frameworks, methods, applications, limitations and future directions without systematically reviewing the literature or narratively chronologizing the historical literature.

Objectives of the study

The specific objectives of this review are to:

- □ Classify major types and sources of uncertainty
- □ Review mathematical frameworks for uncertainty representation
- □ Examine statistical methods for uncertainty quantification
- □ Discuss uncertainty propagation methods and complex-system applications
- □ Compare strengths, limitations, and best-use cases of major approaches
- □ Identify current challenges and future research directions

This article includes some of the conceptual, mathematical, statistical, computational and

applied aspects of uncertainty modelling. It covers, among others, probability theory, fuzzy sets, interval methods, evidence theory, Bayesian inference, Monte Carlo simulation, sensitivity analysis, stochastic modeling, robust optimization and uncertainty quantification using machine learning. Examples of application areas include engineering, finance, climate modelling, healthcare, artificial intelligence and supply chain systems. This paper is intended to be a comprehensive review and does not employ a formal search protocol or screening procedure, nor a chronological narrative structure. It does not do this, however; rather, it sorts the literature by topic, to facilitate cross-disciplinary understanding and applied choice of methodological approach. (Aven, 2016).

2. Methodology

The methodology used in this review is comprehensive, which involves conceptual and thematic synthesis. The literature used was chosen from the cornerstone books, seminal theoretical

works and latest peer-reviewed works that explored the uncertainty modeling. These varied from probability and statistics literature, to engineering reliability and risk analysis studies, to computational modelling and simulation studies, to machine learning and artificial intelligence studies and to applied domain-specific literature. The criteria for including the focus on the theory were: theoretical relevance, methodological importance, cross-disciplinary applicability, and recent advances in uncertainty quantification. The material reviewed was arranged according to concepts, mathematics and statistics, uncertainty propagation methods, complex-system modeling, application areas, comparative evaluation, challenges, and future research directions, which allowed for an integrated view of the uncertainty modeling approaches, both in methodological and application contexts. Figure 1 illustrates the methodology used to review the literature, and the selection, classification and synthesis of its content into a thematic framework for analysis of approaches to modelling uncertainties.

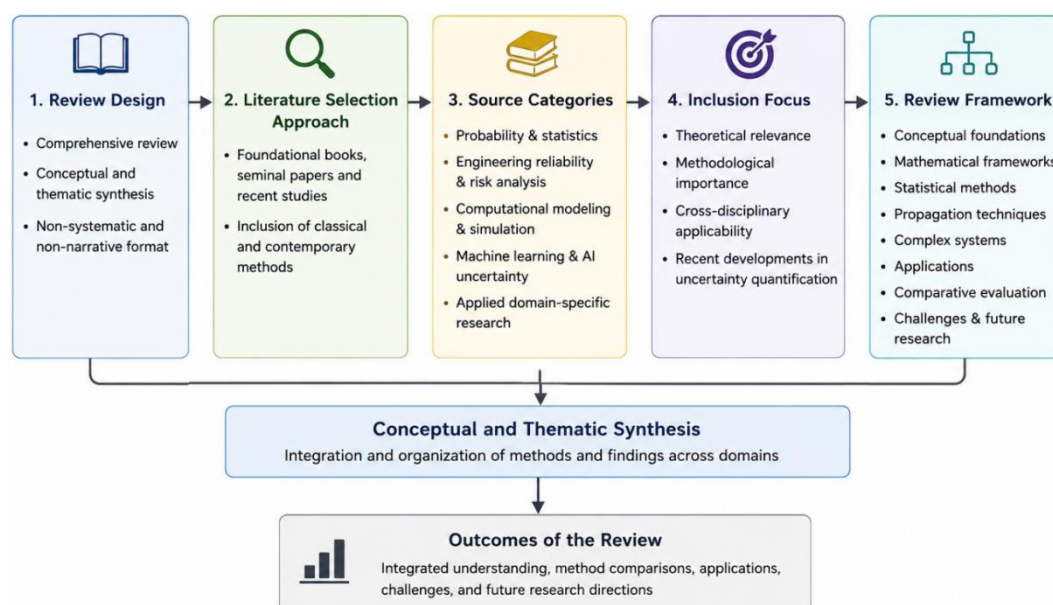


Figure 1. Methodological Framework for the Comprehensive Review

3. Results

3.1 Conceptual Foundations of Uncertainty

Uncertainty is the lack of completeness or

perfection of the knowledge of an event, a parameter, a model, a system or an outcome. Classical decision theory makes a distinction between uncertainty and risk, with the

understanding that risk can be measured as a probability, but uncertainty cannot be so measured, or can only be measured as a very poorly defined probability. (Knight, 1921). Also, uncertainty in applied modelling can involve the notions of ambiguity, variability and ignorance, which have a bearing on interpreting evidence and on making decisions (Morgan et al., 1990).

3.1.1 Definitions and Classifications

The uncertainty can be due to natural randomness and inherent variability in the system (aleatory uncertainty) or due to lack of knowledge (epistemic uncertainty), which can be reduced by more knowledge (better data, models, or expert judgment) (Der Kiureghian & Ditlevsen, 2009). Ontological uncertainty is whether the system structures are known or whether conditions or situations are unforeseen. Model uncertainty is the difference between the mathematical assumptions, structure and/or simplifications and the reality. (Oberkampf et al., 2002). Parameter uncertainty is

due to unknown or estimated data inputs, and measurement uncertainty is due to the observational error, limitations of instruments and noise in data.

3.1.2 Sources of Uncertainty

Uncertainties are primarily due to data limitations, random variation, lack of knowledge, model assumptions, and expert judgments. Data limitations include small samples, missing data, bias and/or poor data measurements. Randomness and noise are a consequence of natural variations of physical, biological, economic and social systems. Epistemic uncertainty is common with incomplete knowledge, particularly for complex systems and systems that are poorly observed. (Walker et al., 2003). Expert judgment will be helpful when empirical data are lacking, but it can lead to subjectivity and inconsistencies. (Helton et al., 2004). The most significant types of uncertainties and their implications are summarised in Table 1.

Table 1. Conceptual Classification of Uncertainty in Modeling and Decision-Making

Uncertainty Type	Meaning	Typical Source
Aleatory	Natural randomness	Variability and noise
Epistemic	Lack of knowledge	Limited data
Model	Imperfect representation	Assumptions
Parameter	Unknown input values	Estimation error
Measurement	Observation error	Instruments and sampling

3.2 Mathematical Frameworks for Modeling Uncertainty

The mathematical frameworks aim to be able to represent different types of uncertainty in scientific and engineering problems. The uncertainty may be due to randomness, lack of knowledge, or imprecise observations, and the framework to be used will depend on the type of uncertainty involved. These methods can be used to represent, analyze and make decisions in an uncertain situation in a systematic manner.

(Kolmogorov, 1956).

Probability theory is the most mature framework, and the uncertainty will be modeled by random variables and probability distributions, conditional probability, and dependence relations. In the classical and frequentist view, the focus is on the frequency of an observation; in the Bayesian view, the focus is on the degree of belief in something based on evidence. (Kolmogorov, 1956). This makes the theory of probability the basis of statistical inference, risk analysis and predictive modelling. (Jaynes, 2003).

Fuzzy set theory is used to deal with the uncertainty due to the presence of vagueness and

not randomness. It is designed to have membership functions that represent partial truth, allowing it to be used in linguistic uncertainties like “high risk,” “low reliability,” or “moderate confidence” (Zadeh, 1965). Fuzzy inference systems are applicable where expert information is available, but it is hard to specify exact numbers in terms of probabilities.

Interval analysis is a method of representing uncertainty in terms of intervals and not with exact values. This technique is valid when ranges of parameters are only known, in particular for robust computation, engineering models. (Moore, 1966). Incorporating imprecise probability further generalises this concept by using a set of probabilities rather than a single probability distribution. (Walley, 1991).

Evidence theory, especially Dempster–Shafer theory, is a theory that can be used for reasoning

with incomplete or conflicting evidence. It is based on belief functions and plausibility functions to express the level of support of statements, where only partial evidence is available. (Shafer, 2020).

Possibility theory is an ally of fuzzy logic that relies on possibility and necessity measures as a means of incomplete knowledge modeling, and ranking a set of plausible outcomes. (Dubois et al., 1991). Possibility theory allows possibility degrees to be anything other than summing to one, and can be used in incomplete or qualitative information. (Dubois & Prade, 1997) Unlike probability.

Table 2 compares the practical applicability of these frameworks by outlining their typical use cases and key limitations in real-world modeling scenarios.

Table 2. Practical Suitability of Mathematical Uncertainty Frameworks

Framework	Detailed Practical Suitability
Probability theory	Suitable for modeling random uncertainty when sufficient empirical data or reliable probability distributions are available. It is widely used in risk analysis, statistical inference, reliability assessment, forecasting, and decision-making under measurable uncertainty.
Fuzzy set theory	Suitable for representing vague, linguistic, or qualitative uncertainty where exact numerical probabilities are difficult to assign. It is useful in expert systems, decision support, control systems, and human judgment-based modeling.
Interval analysis	Suitable when uncertain values can only be expressed as lower and upper bounds. It is useful in engineering design, robust computation, safety analysis, and systems where parameter ranges are known but exact distributions are unavailable.
Evidence theory	Suitable for combining incomplete, uncertain, or conflicting evidence from multiple sources. It is commonly applied in sensor fusion, diagnostic systems, reliability assessment, and decision-making under partial information.
Possibility theory	Suitable for modeling incomplete knowledge and plausible outcomes when probabilistic information is insufficient. It is useful in qualitative reasoning, expert judgment, fuzzy systems, and uncertainty representation under limited data conditions.

Figure 2 illustrates how different mathematical frameworks correspond to different forms of uncertainty.

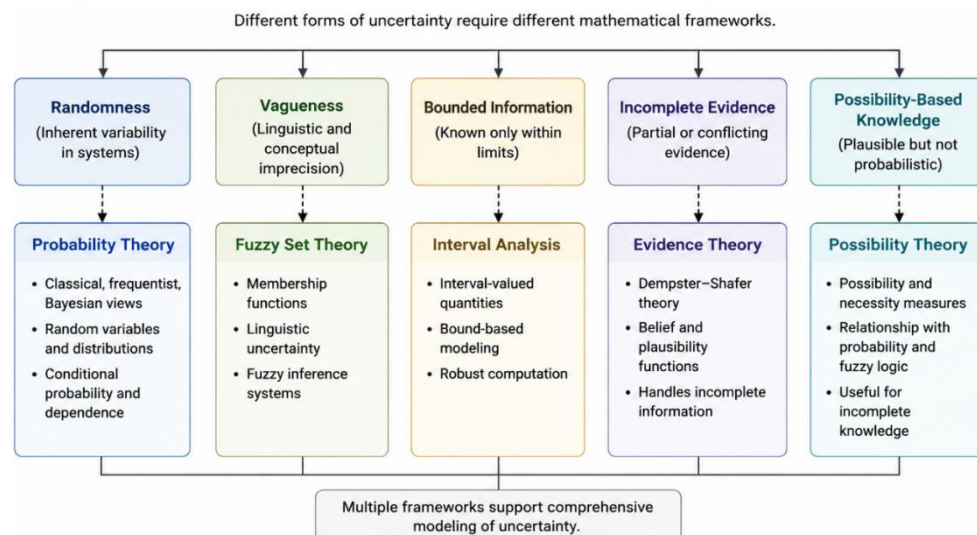


Figure 2. Mathematical Frameworks for Different Forms of Uncertainty

3.3 Statistical Methods for Uncertainty Quantification

Using statistical methods provides a quantification of the uncertainty in the data, parameters, models and predictions derived from incomplete information. These methods offer a structured way of analysing the variability, updating the knowledge and gaining insight into the impact of the uncertainty in the input on the output of the model.

3.3.1 Inferential Statistics

The basic tools used to make inferences from data collected from a sample are called inferential statistics. The variability in sampling is due to the difference between the estimates for the sample and the true values for the population. The confidence intervals are used to provide an estimate of the uncertainty of the estimation of an unknown parameter, and are used to express the likely range of that parameter. Hypothesis testing is used to assess the validity of the assumption that the data adhere to the statistical model, and aid in decision-making (Ziegel, 2002).

3.3.2 Bayesian Inference

Bayesian inference is the ability to combine prior distributions of some quantities of interest with observed data using likelihood functions to create the posterior distributions. These posterior distributions reflect a new uncertainty in light of the new evidence. Bayesian techniques are

especially useful when dealing with complex systems, when using expert-elicitation for the construction of models, and when updating simulation uncertainty (Gelman et al., 1995). If there is any uncertainty in the parameters of the model, or there are discrepancies between it and the experimental observations, these can be taken into account simultaneously in a model calibration using a Bayesian approach (Kennedy & O'Hagan, 2001), which leads to better predictions. Bayesian analysis is also useful when there is a shortage of experiments or the experiments are expensive (O'Hagan, 2006).

3.3.3 Monte Carlo Simulation

Monte Carlo simulation is a repeated random sampling to quantify uncertainty. Well used when there are no analytical solutions or when they are very hard to obtain. Monte Carlo is an approach of generating many input scenarios; it is used to approximate the resulting output distributions and for the propagation of uncertainty in a probabilistic fashion (Robert & Casella, 2004). Variance reduction techniques are used to help improve the computational efficiency by reducing the simulation error without an appreciable increase in computational cost. (Fishman, 2013).

3.3.4 Bootstrap and Resampling

Bootstrap and resampling methods are done by repeatedly sampling from observed datasets to estimate the uncertainty. These methods are

especially helpful if the sampling distribution is not known or is a complicated distribution. Bootstrap confidence intervals are non-parametric, flexible estimates of uncertainty, and are a popular means to validate models and assess model performance. (Efron & Tibshirani, 1993).

3.3.5 Sensitivity Analysis

Sensitivity analysis is the analysis of the variability of outputs of a model based upon the variability of the inputs. Local sensitivity analysis involves analysing the variations in some parameters around fixed values, and global sensitivity analysis

involves studying the variations in the parameters in the complete input space. Variance-based decomposition techniques quantify the contribution of the different input variables to the uncertainty in the output. (Saltelli et al., 2008). The polynomial chaos-based methods also enable the propagation of the uncertainty in computational models efficiently. (Sudret, 2008). An integrated overview of the main statistical methods is given in Table 3, which shows their functions, outputs, strengths and limitations regarding uncertainty quantification.

Table 3. Overview of Statistical Methods for Uncertainty Quantification

Method	Main Function	Typical Output	Strength	Limitation
Inferential statistics	Estimates population properties	Confidence intervals	Simple and interpretable	Depends on assumptions
Bayesian inference	Updates uncertainty with evidence	Posterior distributions	Incorporates prior knowledge	Sensitive to prior selection
Monte Carlo simulation	Propagates input uncertainty	Output distributions	Flexible for complex systems	Computationally expensive
Bootstrap resampling	Estimates sampling uncertainty	Resampled intervals	Non-parametric and practical	Requires representative data
Sensitivity analysis	Identifies influential inputs	Sensitivity indices	Supports model understanding	Costly in high-dimensional cases

3.4 Uncertainty Propagation Methods

Uncertainty propagation is a way of looking at the uncertainty with respect to the input and assumptions/parameters for a model and how these uncertainties affect the model output. Analytical propagation is a mathematical approach which directly provides an estimation of the uncertainty of the output and is only applicable to simple or linear systems. Especially for more complex models, for which closed-form solutions cannot be obtained, there are computational schemes of propagation, called numerical propagation. The uncertainty in the output can be estimated by using distributions and sampling in probabilistic propagation, and the responses of

stochastic models can be efficiently represented by polynomial chaos methods. (R. G. Ghanem & Spanos, 2003). These techniques are especially interesting for computational sciences, where they can be applied to decrease the number of simulations to be carried out. (Xiu, 2005). Fuzzy propagation is used when uncertainty is not a probabilistic one, but is a vague or linguistic type of uncertainty.

The hybrid uncertainty propagation method is a combination of the different uncertainty propagation methods (probabilistic, fuzzy, interval, or sensitivity) to account for both aleatory and epistemic uncertainty. In complex systems, the propagation of errors can be the

result of uncertainty in data inputs, numerical approximation, model assumptions and interactions between model parameters. Propagation studies are complemented by sensitivity analysis, which can identify which parameters are most important in affecting the model outputs, given the uncertainty of the model parameters. (Saltelli et al., 2008). To address the

uncertainty, therefore, flexible propagation strategies are needed that can be used in all of these contexts: analytical, simulation-based, and hybrid modeling (Sullivan, 2015). As shown in Figure 3, the basic pathway of the transfer of uncertain inputs to a model involves the expression of the uncertain outputs.

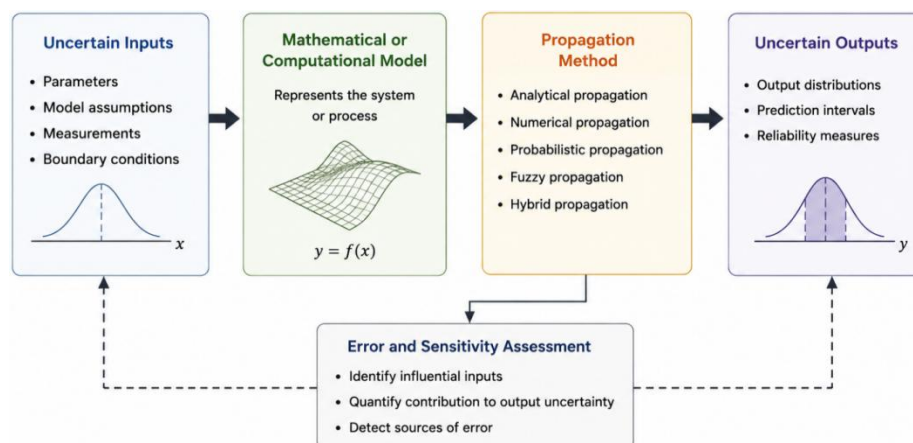


Figure 3. General Framework for Uncertainty Propagation

As indicated in the figure 3, the propagation of uncertainties can be applied beyond the estimation of the output. It also facilitates sensitivity and error diagnosis, as well as a better understanding of the behaviour of complex models in relation to uncertainty.

3.5 Modeling Uncertainty in Complex Systems

When modelling uncertainty in complex systems, it is necessary to combine stochastic representations, optimisation techniques and

3.5.1 Stochastic Models

Stochastic models are mathematically based models of a complex system that allow for randomness and variability in time and/or space. A stochastic process is a process that is defined as:

$$\{X(t), t \in T\} \quad (1)$$

where $X(t)$ is the random state at time t of the system. This formulation can be used to model systems that have an inherent randomness, including financial markets, biological processes and engineering systems. (Ross, 2014). A

machine learning procedures to capture the uncertainty, incomplete knowledge and dynamic interactions. The uncertainty modelling is a critical element to ensure reliable prediction and decision making in complex systems, which have potentially a very high number of dimensions, nonlinear behaviour, and changing system structure. Stochastic models, robust and stochastic optimization and data-driven learning frameworks explicitly are used to model uncertainty in these system

fundamental class of stochastic processes are the Markov chains, which have the memoryless property:

$$\begin{aligned} &P(X_{n+1} \\ &= x \mid X_n, X_{n-1}, \dots, X_0) \\ &= P(X_{n+1} \\ &= x \mid X_n) \end{aligned} \quad (2)$$

This is a useful property when modelling because it means that the next state of a Markov chain only relies on the current state and not on the history of the chain, such as in reliability analysis, queuing systems and decision

processes. Brownian motion (also known as Wiener process) is a continuous time randomness and is defined by:

$$W(0) = 0, \quad W(t) - W(s) \sim N(0, t - s) \quad (3)$$

where changes are normally distributed around zero with a variance that is proportional to time. Random fields are used to model in the spatial domain:

$$\{X(s), s \in D\} \quad (4)$$

Here are the spatial coordinates. They are popular engineering and environmental system models that model spatial variability. (R. G. Ghanem & Spanos, 2003).

3.5.2 Robust and Stochastic Optimisation

The goal of optimisation under uncertainty is to find decisions that are good under a variety of uncertainties. In the case of robust optimization it is about the worst case and is formulated as:

$$\min_x \max_{\xi \in U} f(x, \xi) \quad (5)$$

where x 's are the decision variables and ξ 's are uncertain parameters in an uncertainty set, U . The mean function $m(x)$ and covariance kernel $k(x, x')$ can be used to make predictions with uncertainty (Williams & Rasmussen, 2006). The Bayesian NN is an extension of deep learning, which introduces probability distribution over the model parameters, and can estimate uncertainty in the prediction (Abdar et al., 2021). The ensemble learning method is used to obtain as many models as possible to increase the accuracy of prediction and to compute uncertainty by using the variations in prediction. Moreover, uncertainty estimation is further advanced by deep learning uncertainty methods that involve the use of dropout, probabilistic architectures, etc., in high-dimensional data environments (Yang et al., 2021). However, there are ways to combine the physical knowledge of the system with machine learning, such as scientific machine learning, which aims to enhance the reliability and interpretability of machine learning models in complex systems (Psaros et al., 2023).

This way, they can be solutions under extreme conditions. Stochastic optimisation takes into account probability distributions of uncertain parameters and optimises decisions based on the expected values. Chance-constrained optimisation is a further extension of this due to the fact that it requires that the constraints are satisfied with a desired probability.

$$P(g(x, \xi) \leq 0) \geq 1 - \alpha \quad (6)$$

with an acceptable risk level, α . These are important techniques in engineering design, finance and operations research that are directly relevant to the performance of systems in the presence of uncertainty.

3.5.3 Machine Learning-Based Uncertainty Modeling

Machine learning methods offer general methods and tools for modelling the predictive uncertainty of complex systems. The prediction using Gaussian processes is a distribution over functions:

$$f(x) \sim GP(m(x), k(x, x')) \quad (7)$$

3.6 Applied Domains

3.6.1 Engineering and Reliability Analysis

Uncertainty modelling is at the heart of engineering design, the safety of structures and reliability assessment. The loads on an engineering system, the properties of its materials, the degradation that may occur, and the environmental conditions may be uncertain. Reliability analysis finds the probability of failure and performance of the system in the presence of uncertainties, with the use of probabilistic and statistical methods. (Melchers & Beck, 2018). Additionally, uncertainty quantification, verification, and validation provide confidence in the computational engineering model, detect the various types of errors (numerical, parameter, and model-form) and reduce these errors. (Roy & Oberkampf, 2011).

3.6.2 Finance and Risk Management

Uncertainties are modelled in finance to estimate the market risk, credit risk, losses of a

portfolio and extreme events. Probability distributions, dependence models, value-at-risk (VaR), stress testing, and scenario analysis are the key components of quantitative risk management that are used to assess financial exposure. (McNeil et al., 2015). Volatility, nonlinear dependence and rare but high-impact events are all important in these methods since they apply to financial systems.

3.6.3 Climate and Environmental Modeling

The uncertainty in the climate and environmental models is due to emissions scenarios, to the physics of the model, to parameter estimation, to observational limitations and to the model structure. Uncertainty quantification is used to assess the environmental risk and policy-relevant prediction, as well as the climate projection. Careful communication of uncertainty is needed in climate modeling given the long time horizons and lack of scientific knowledge on which decisions are frequently made. (Legg, 2021).

3.6.4 Healthcare and Epidemiology

In health care and epidemiology, uncertainty impacts disease modelling, evaluation of treatment, economic evaluation and public health decisions. Clinical and policy findings are subject to parameter uncertainty, sampling variability and model assumptions. A widely used method of sensitivity analysis is the probabilistic approach, which can be used to assess the impact of uncertain input parameters on health-economic outcomes. (Briggs et al., 2012).

3.6.5 Artificial Intelligence and Decision Support

Uncertainty-aware predictions are becoming more and more desired in the development of Artificial Intelligence (AI) systems in order to enhance trust, reliability and decision support. Deep learning models can be confident in their prediction of incorrect answers, particularly when there's a change in distribution or in the lack of data. Uncertainty quantification in AI

can help to differentiate between aleatory and epistemic uncertainty and make the deployment of AI in high-stakes applications safer. (Abdar et al., 2021). Scientific machine learning also introduces physical constraints to data-driven learning to incorporate uncertainty when modeling. (Psaros et al., 2023).

3.6.6 Supply Chain and Operations Research

Demand uncertainty, lead-time variability, disruption risks, transportation delays and supplier instability are some of those uncertainties that affect the supply chain systems. Optimisation models in the presence of uncertainty and risk enable the resilient planning, inventory management, routing and resource allocation. These are useful to enhance decision-making through the assessment of uncertainty in the operational environment and resilience strategies. (Suryawanshi & Dutta, 2022).

4. Discussion

4.1 Comparative Analysis of Uncertainty Modeling Approaches

Different uncertainty modelling approaches are based on different assumptions, require different types of data, have different demands for computing resources and have different strengths in interpretation. The probability theory is rigorous and widely accepted for random uncertainty, and the Bayesian methods are an extension of the theory of probability to update beliefs in the light of evidence. (Jaynes, 2003; Kolmogorov, 1956). Vagueness, linguistic uncertainty (Zadeh, 1965) It is more appropriate to be addressed using fuzzy logic and possibility theory. Interval and imprecise probability approaches are appropriate when only limited and/or bounded information is available. (Moore, 1966; Walley, 1991). Evidence theory can be used to reason about the incomplete evidence, but it might be computationally costly in complex systems. (Shafer, 2020). In general, there is no "best" method, and the choice of method should be based on the nature of the uncertainty, quality of the data, size of the model, and the nature of the decision. Table 4

gives an overview and comparison of the main uncertainty modelling approaches with emphasis

on their advantages, disadvantages and areas of application.

Table 4. Comparative Analysis of Major Uncertainty Modeling Approaches

Approach	Strengths	Limitations	Best Use Cases
Probability theory	Rigorous and accepted	Requires reliable data	Risk analysis
Bayesian methods	Updates with evidence	Prior sensitivity	Decision-making
Fuzzy logic	Handles vagueness	Lower statistical rigor	Expert systems
Interval analysis	Simple and robust	Conservative bounds	Engineering systems
Evidence theory	Handles incomplete evidence	Computationally intensive	Sensor fusion
Monte Carlo simulation	Flexible and generalizable	Computationally expensive	Complex models
Machine learning methods	Scalable and adaptive	Interpretability challenges	Prediction systems

4.2 Theoretical Implications

The theoretical consequences of uncertainty modelling are that integrated taxonomies are needed that make a clear distinction between risk, uncertainty, variability, ignorance and ambiguity. Classical notions of measurable risk and ultimate uncertainty still have relevance in the choice of appropriate modelling methods. (Knight, 1921). But uncertainty is not just one-dimensional and context dependent; there are also broader classifications which relate the conceptual definition of uncertainty with practical modeling needs. (Walker et al., 2003). Implications are that probabilistic and non-probabilistic models are not competing models, but should be considered complementary. Randomness can be captured by a probabilistic approach, and the vagueness, bounded knowledge, and incomplete evidence by fuzzy, interval and evidence-based approaches. The differentiation of aleatory and epistemic uncertainty is important, particularly because the aleatory uncertainty is inherent variability which cannot be reduced by better knowledge, while the epistemic uncertainty can be reduced by better knowledge. (Der Kiureghian & Ditlevsen, 2009).

This separation facilitates clearer uncertainty communication, more robust interpretation of the models and improved decision-making under complex circumstances. (Aven, 2016).

4.3 Practical Implications

When choosing an approach for modeling uncertainty, the data available, the complexity of the model, computational resources, and the purpose of the decision context should be taken into account. When information is well known, it is suitable to use probabilistic methods, while in cases of partial information or lack of it, it is better to use interval or fuzzy and/or Bayesian methods. (Smith, 2024). The selection of method directly impacts the level of prediction accuracy, the assessment of reliability, as well as making decisions under risk in engineering, finance, healthcare, operational planning, etc. (Melchers & Beck, 2018). Hybrid uncertainty models have proven to be useful due to a number of real-world problems that involve both inherent randomness and knowledge-based uncertainty. The use of a combination of probabilistic, statistical and machine learning methods can enhance the

robustness of predictions and provide better decision support in complex scenarios. (Nemani et al., 2023). Moreover, in practice, computational efficiency is an important factor to consider, especially for large-scale simulations and optimisation problems, where uncertainty quantification needs to be fast and cheap. (Sullivan, 2015).

4.4 Key Challenges

Several challenges have been identified to date that affect the reliability and applicability of uncertainty modeling in the realm of complex models involving many interacting variables, parameters and assumptions (high-dimensional uncertainty), which leads to higher computational demands and uncertainty sensitivity analysis.

From a practical perspective, a major concern is that if the mathematical models are too simple to capture the behaviour of the actual system, then the assumptions of model misspecification will be violated, and while some uncertainty may be reduced by adopting a Bayesian approach to calibration, assumptions about the discrepancy, along with assumptions about model structure, are still required (Kennedy & O'Hagan, 2001).

When simulations are costly, nonlinear or have limited experimental data, the analysis of the computer models becomes more difficult, and estimating uncertainty becomes more challenging (O'Hagan, 2006).

Uncertainty estimates are compromised by poor quality of data, missing data and measurement errors and V&V and UQ are essential to enhance model credibility and reproducibility (Roy & Oberkampf, 2011).

Interpretability is another challenge of machine learning, as uncertainty estimates are hard to explain in high-dimensional models, and the combination of aleatory and epistemic uncertainty in a single framework is still a significant issue in scientific machine learning and complex-system prediction (Abdar et al., 2021; Psaros et al., 2023).

4.5 Emerging Trends

Hybrid, explainable and data-driven approaches

are more and more influencing emerging trends in uncertainty modeling. Hybrid uncertainty modeling is a combination of the probabilistic, Bayesian, fuzzy, interval and machine learning approaches, which can be used to model multiple types of uncertainty in one approach. The probabilistic machine learning paradigm is also growing due to its ability to give an estimation of uncertainty along with the output, which makes the estimation reliable in such complex applications ((Abdar et al., 2021). Physics-informed uncertainty models also fuse the laws of scientific and engineering systems with data-driven learning, which helps achieve more trustworthy predictions of scientific and engineering systems. (Yang et al., 2021). Digital twins and real-time uncertainty updating are emerging as key technologies in the context of monitoring, prediction and adaptive decision-making of dynamic systems. Efficient surrogate models and scalable uncertainty quantification approaches are needed for these approaches. (Gramacy, 2020). The fields of explainable uncertainty quantification and uncertainty-aware AI are also becoming popular, as it is crucial to be able to articulate uncertainty in a reliable manner for ethical, transparent and policy-relevant decisions (Aven, 2016).

5. Conclusion

Mathematical and statistical approaches to uncertainty modeling play a critical role in understanding, quantifying, and managing uncertainty in complex systems. In this review, it is emphasised that uncertainty can be attributed to a number of sources, such as the inherent randomness, lack of or imperfect knowledge, model assumptions, parameter estimation errors, measurement limitations and computational constraints. Consequently, a single modelling approach is not able to incorporate all the uncertainty features, and flexible and context-specific approaches should be used. An important feature of this article is that it brings together all the above, from conceptual basis, mathematical framework, statistical methods, uncertainty propagation methods and applications, in one place, across a variety of fields. The synthesis of

these elements offers a coherent view on the topics, enabling both researchers and practitioners to decide on which methods to use in accordance with the nature of the uncertainty, available data, and the decisions to be made. This integrated view also highlights the need to integrate various modelling methods and approaches to enhance robustness and reliability. In the future, research should focus on the design of scalable hybrid models which are able to simultaneously deal with both aleatory and epistemic uncertainty. But there is also a need for artificial intelligence systems that are interpretable and that can produce

uncertainty-aware systems for transparent decision-making. Further attention needs to be paid to the standardisation of validation approaches, on-the-fly uncertainty representation of digital twin systems, and to the creation of cross-disciplinary modeling standards. Finally, to further advance the modeling of uncertainties, it is crucial to consider a harmonious combination of hybridisation, interpretability, scalability, and practical applicability of the models for improved decision-making in various scientific, engineering, economic, healthcare, environmental and AI-driven systems.

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