

How the handmade patterns can develop a real intelligence in the 3D geometric thinking of the learners?

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This paper investigates the teaching and learning of third-degree remarkable identities at the middle school level, with particular attention to students' transition from second-degree to third-degree generalizations. The study aims to analyze the errors and difficulties encountered by students when extending the identity $(a+b)^2$ to $(a+b)^3$ focusing on the coordination between geometric and algebraic reasoning. The research is based on a didactic intervention integrating symbolic expansion and volumetric modeling through the manipulation of cardboard cubes representing the decomposition of a cube of edge length $(a + b)$. The analysis of students' productions reveals conceptual misunderstandings, insufficient combinatorial reasoning, and significant difficulties in coordinating semiotic representation registers.

The results indicate that many observed errors stem from the overgeneralization of two-dimensional formulas and from disruptions in the articulation between geometric and symbolic registers. These difficulties reflect deeper epistemological and didactic obstacles that hinder the construction of structural understanding of remarkable identities in three dimensions. The findings highlight the importance of embodied and spatial mediation in supporting the development of algebraic thinking.

Keywords: Algebraic thinking, geometric thinking, mathematics education, ICT, handmade patterns, remarkable identities of degree three.

1. Introduction.

The teaching of school algebra frequently encounters difficulties related to the

discontinuity between the modes of thinking mobilized in geometry and those required for algebraic work. This discontinuity cannot be

reduced to the mere introduction of new symbols or computational techniques; rather, it involves a profound transformation in students' relationship to mathematical objects, which shift from perceptible and manipulable entities to abstract formal expressions. As several studies have shown, the transition from arithmetic or geometric reasoning to algebraic generalization requires a restructuring of cognitive and semiotic processes rather than a simple accumulation of procedural knowledge (Duval, 2006; Kieran, 2007; Radford, 2014).

It appears that the mathematical context in which students evolve may contribute significantly to the development of algebraic thinking. According to Teppo and Esty (1995), the structure of a task influences the type of knowledge it mobilizes. In mathematics education research, this transition is widely identified as a major issue in the construction of meaning for algebraic concepts. Thus, the teacher's role is to support students in their progression toward a structural conception of algebra, that is, toward genuine algebraic thinking (Labelle, 2008).

Research in mathematics education consistently emphasizes that the structure of tasks strongly influences the type of reasoning mobilized by students (Kaput, 2008; Teppo & Esty, 1995). When algebra is introduced without meaningful connections to prior geometric or numerical experiences, students often adopt procedural strategies detached from conceptual understanding. This phenomenon has been widely documented in studies on algebraic generalization and symbolic manipulation (Carraher, Schliemann, & Schwartz, 2008; Radford, 2014).

Remarkable identities provide a particularly fruitful context for studying this transition. Although their validity is purely algebraic, their introduction in schools is often disconnected from conceptually accessible justifications for students, frequently resulting in procedural learning based on memorization. While geometric interpretations of two-dimensional remarkable identities are relatively

well established in teaching practices, three-dimensional interpretations—such as the expansion of $(a + b)^3$ —remain marginal, even though they offer rich opportunities to connect spatial reasoning, decomposition, and symbolic expression.

In this study, we hypothesize that engaging students in a concrete manipulation task based on the study of a cardboard cube with side length $a + b$ can support the gradual transition from geometric thinking to meaningful algebraic reasoning. Decomposing the cube into sub-cubes and rectangular prisms allows students to identify the different volumetric contributions associated with the terms a^3 , $3a^2b$, $3ab^2$, and b^3 before any explicit symbolic formalization. In this way, the remarkable identity emerges as the result of spatial organization rather than as an arbitrary rule.

This situation is designed as a didactic environment in the sense of the Theory of Didactic Situations (Brousseau, 1997), in which students are prompted to act, formulate, and validate relationships while interacting with tangible material. The goal is not only to understand the remarkable identity itself but also to study the reasoning processes students engage in when transitioning from a three-dimensional geometric register to a symbolic algebraic register. Attention is given to the obstacles encountered, the invariants perceived, and the transformations of representations enacted during the activity (Duval, 2006).

The aim of this research is therefore to analyze the didactic potential of a volumetric manipulation situation to support the genesis of algebraic understanding of three-dimensional remarkable identities, and more broadly, to shed light on the conditions that favor the transition from geometric thinking to algebraic thinking in secondary mathematics education.

2. The theoretical framework.

To develop algebraic thinking, students observe patterns arising from various situations and represented in different semiotic forms, such as drawings, tables of values, and graphs.

Algebra is formally introduced in Cycle 4 of lower secondary education. According to the Ministry of National Education and Youth (MENJ, 2018), algebraic knowledge is taught to students aged 13 to 15 with the objective of developing broad mathematical competencies:

- **Represent:** Choose and coordinate appropriate frameworks (numerical, algebraic, geometric) to solve a problem or study a mathematical object.
- **Calculate:** Perform calculations using algebraic language (letters, symbols, and formal expressions).
- **Communicate:** Establish connections between natural language and algebraic language, and distinguish the specific features of mathematical language from those of everyday French language (Squalli, 2020).

These competencies highlight that algebra learning is not limited to symbol manipulation but involves the coordination of multiple representations and the progressive structuring of mathematical meaning.

Algebra appears essentially as a linguistic tool and does not seem able to exist as a mathematical field without the use of letters. However, the formal introduction of algebra can be prepared in earlier years by fostering algebraic thinking “before the letter.” This didactic orientation has notably been adopted in the Quebec curriculum (Squalli, 2020).

According to Sierpiska (1999), algebra is a product of analytical thinking and requires a specific mode of cognitive functioning. It is grounded in external and conventional systems of representation, enabling individuals to engage with mathematical objects through symbolic and verbal descriptions. In this perspective, language is not merely used to name objects but to describe and, above all, to create them through definitions and systems of propositions (Labelle, 2008).

Algebra constitutes a specific field of mathematical activity, and algebraic thinking refers to the mathematical mode of reasoning characteristic of this activity. Radford (2018)

defines algebraic thinking as involving indeterminate quantities treated analytically and culturally developed modes of symbolizing these quantities and their operations. Importantly, algebraic thinking can be mobilized without necessarily using literal symbolism, provided that reasoning engages with generality, structure, and indeterminacy.

At an operational level, algebraic thinking manifests through specific reasoning tendencies: interpreting equality as a relation of equivalence, leaving operations open-ended, symbolizing and operating on symbols, and adopting a structural perspective—for instance, seeing a numerical expression as an object rather than merely a sequence of calculations (Squalli, 2020).

A brief analysis of curriculum concepts and processes reveals numerous opportunities for fostering algebraic thinking: the study of sequences and patterns from early grades, the exploration of numerical properties (even/odd, prime, composite, multiples), additive and multiplicative structures, divisibility criteria, equality and equations, equivalent expressions, and order relations. The development of reasoning skills, particularly observing regularities, classifying mathematical objects, and justifying mathematical statements, also contributes significantly to the emergence of algebraic thinking.

Squalli (2000) identifies four essential competencies underlying algebraic thinking:

1. The ability to think analytically;
2. The ability to construct, interpret, and validate algebraic models of real or mathematical situations;
3. The ability to manipulate algebraic expressions according to established rules;
4. The ability to abstract and generalize relationships, structures, and rules.

These competencies may be interpreted cognitively as discursive tendencies shaping classroom interactions. Thus, in environments that aim to develop algebraic thinking, classroom discourse should explicitly value

generalization, structural reasoning, and analytical justification.

Also, a progression in the development of algebraic thinking from kindergarten to 12th grade. The study of patterns, functions, and algebra should begin informally in kindergarten and progress in sophistication and depth throughout the curriculum. It appears that these authors consider the period from kindergarten to Grade 2 as a preliminary phase of exploration of algebraic concepts and development of a language to represent and describe generalities (rules, regularities, etc.).

From grades 3 to 5, according to the authors, students develop algebraic ideas when they construct or identify numerical or geometric patterns, describe patterns verbally and represent them using tables or symbols, seek and apply relationships between quantities that vary to make predictions, develop generalizations (or rules) and test them, use graphs to describe patterns and make predictions, and explore numerical properties. Students also explore the concept of variables and begin to use them as placeholders to represent unknowns or quantities that vary. At the end of this level, students are encouraged to use letters to represent variables in a generalization context. (Squalli H. , 2020)

Generalization and analyticity are widely recognized as core characteristics of algebraic thinking (Lee, 1996; Kaput, 2008; Carraher et al., 2008; Radford, 2010, 2014; Squalli, 2000; Grugeon, 1995). Lee (1996) argues that generalization activities constitute the primary gateway to algebraic culture, noting that functions, modeling, and problem solving are all fundamentally forms of generalization.

Grugeon (2000) further highlights the relationship between generalization and algebraic competence, showing how generalization problems bridge numerical and algebraic knowledge. Algebraic language enables systematic formulation and resolution of problems in their general form.

Interest in generalization has led to the design of didactic situations based on varied representations and modeling contexts to promote algebraic thinking (Bronner, 2021).

Students are encouraged to use their geometric thinking and spatial awareness in their daily activities and in various disciplinary or interdisciplinary contexts. The van Hiele model (van Hiele, 1984) distinguishes five levels in the development of geometric thinking in children:

- **Level 0 (visualization):** figures are identified solely by their general appearance.
- **Level 1 (analysis):** children begin to discern the properties of figures but are not yet able to explain them.
- **Level 2 (informal deduction):** children can establish intra- and inter-figure relationships. Definitions make sense, and empirically obtained results are often used in conjunction with deductive techniques.
- **Level 3 (formal deduction):** deduction is perceived as a validation tool within an axiomatic system; the same applies to the respective roles of primitive notions, axioms, definitions, and theorems.
- **Level 4 (rigor):** the pupil (student) is able to place themselves in different axiomatic systems (non-Euclidean geometries, for example) and compare them.

As we can see, levels 0 and 1 are based on “figures” (in the sense of “drawings” [Parzysz 1989]): the corresponding geometry is therefore “concrete” geometry, whose objects are material (drawings, models, everyday objects, etc.), and in which argumentation is based essentially on perceptual criteria. In contrast, the geometry of levels 3 and 4 is “theoretical” geometry, whose objects are conceptual, and in which the only acceptable argumentation is proof.

This leaves level 2, which is, in a way, the pivotal level between these two types of geometry, in which theory is being established in the student—mainly through education—by being constructed against perception, which had been accepted until then. This is the moment when the conflict between what is known and what is

perceived becomes most apparent, which I have already studied in the context of spatial geometry. (Parzysz B., 2001)

Perrin-Glorian (2004) distinguishes between changes of frame and conversions between semiotic representation registers (Duval, 1995). A change of frame involves mobilizing different bodies of knowledge (e.g., geometric, numerical, algebraic), whereas register conversion concerns the coordination of different representations of the same object.

Recent work by Varent (2024), in an analysis of textbooks addressing the area of a rectangle in Grade 5, shows that the transition from geometric tiling to algebraic formulas may generate representational tensions and discontinuities in students' understanding. This analysis highlights the epistemological gap that can arise between geometric configurations and symbolic expressions when the transition is not explicitly supported.

In the case of the volume of a cube, the geometric frame (tiling and decomposition) corresponds to the register of figures. The algebraic frame involves symbolic expressions such as $a \times a \times a$ or $l \times L \times h$, while the numerical frame emerges when formulas are applied in concrete calculations. The coordination of these frames requires students to differentiate between objects and their representations; a fundamental condition for conceptual understanding (Duval, 2006).

In summary, our focus is on the geometric framework associated with the register of figures (tiling), followed by the algebraic framework linked to the symbolic register (formula), and finally the numerical framework connected to the register of numbers (application of the formula). We examine the coordination of representations of geometric objects involved in volume calculation, as well as the transition from the geometric framework to the algebraic and numerical frameworks.

Duval (1995, 2006) emphasizes the necessity of carefully distinguishing mathematical objects from their semiotic representations. This dis-

inction requires a precise analysis of how students move between frameworks and how geometric thinking progressively transforms into algebraic thinking. In line with this perspective, Varent (2024) shows that transitions from geometric configurations to symbolic formulas may generate epistemological and representational tensions when the coordination of registers is not explicitly supported.

3. The problematic. Problem Statement and Research Questions

The diagram of a cube divided into several smaller parallelepipeds is commonly used to visualize the remarkable identity of order three:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

This type of geometric representation is intended to anchor algebraic understanding in spatial intuition.

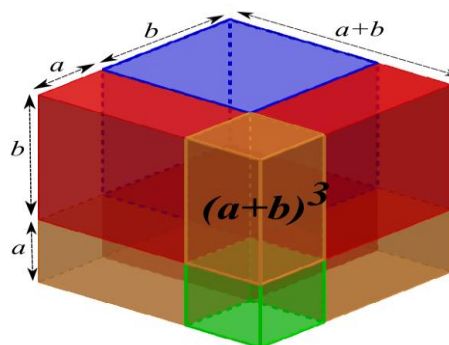


Figure 1 Decomposition of a cube of side length $a+b$

Teaching remarkable identities, particularly the development of $(a + b)^3$, requires both algebraic skills and geometric reasoning when students attempt to make sense of the formula through a spatial approach. Increasingly, teachers use three-dimensional representations, such as the decomposition of a cube, to help students visualize the underlying algebraic structure.

However, classroom observations suggest that this approach does not benefit all students equally. Some experience difficulty in mentally or graphically representing three-dimensional objects, while others struggle to establish meaningful connections between geometric representation and algebraic formulation.

Moreover, generalizing a second-order identity to a third-order identity appears to present additional challenges, especially due to gaps in students' understanding of the cube as a composite volume.

The central research questions guiding this study are therefore:

- What difficulties do students encounter in three-dimensional geometric reasoning when working with the decomposition of a cube?
- What difficulties do students encounter when attempting to generalize a remarkable identity of order three using geometric reasoning?

This issue can be structured around three interrelated areas:

• **Obstacles related to the visualization and representation of three-dimensional objects.**

Do students experience difficulty drawing or mentally representing a 3D shape? What are the possible cognitive or representational causes of such difficulties?

• **Algebraic difficulties related to the expansion and factorization of polynomial expressions.** Do students encounter problems when expanding or factoring expressions? What conceptual or procedural gaps underlie these difficulties?

• **Difficulties in establishing correspondences between geometric objects (cubes, rectangular prisms) and algebraic expressions.** Do students struggle to deduce third-order polynomial formulas from cubic geometric configurations—and conversely, to interpret algebraic expressions geometrically? What factors explain these difficulties?

Teaching remarkable identities constitutes a critical moment in the transition from elementary arithmetic to formal algebra. These identities (e.g., $(a + b)^3$) are frequently introduced as algebraic rules to be memorized, without necessarily ensuring students' conceptual understanding. To address this limitation, several pedagogical approaches

propose the use of geometric models to provide meaning to these identities.

For instance, the expansion of $(a + b)^3$ can be represented by a cube of side length $a + b$, decomposed into smaller cubes and rectangular prisms corresponding to the terms a^3 , b^3 , $3a^2b$, and $3ab^2$. This approach seeks to anchor algebraic understanding in spatial reasoning, based on the assumption that geometric structuring can support algebraic abstraction.

4. The experimentation.

The proposed instructional sequence was organized into two complementary phases. The first, theoretical phase focused on the geometric modeling of the algebraic expansion of $(a + b)^3$. It consisted of three structured tasks designed to support students in coordinating different semiotic registers, particularly the transition from the geometric register to the symbolic register within a three-dimensional (3D) representational framework. The structuring of these tasks aimed to make explicit the representational transformations involved in moving from spatial decomposition to algebraic formulation.

The second, practical phase centered on the material manipulation of $(a + b)^3$, emphasizing the articulation between algebraic abstraction and embodied cognition. Using cardboard cubes, students were guided from two-dimensional (2D) graphical representations to concrete three-dimensional (3D) constructions. This shift was intended to foster the stabilization of meaning associated with the remarkable identity by grounding symbolic expressions in perceptual and spatial experience.

The experimentation was conducted with 173 lower secondary students distributed across four classes from the second and third years of middle school (42 students from class "2/4", 44 from "2/3", 44 from "3/1", and 43 from "3/2"). Four classroom implementations were carried out by the teacher, each structured into two sessions. The first session was devoted to students' written productions based on a

designed worksheet mobilizing prior knowledge of second-degree remarkable identities. The second session involved the physical assembly of cardboard cubes, allowing students to materialize the volumetric decomposition derived from their initial symbolic work.

The study aimed to examine the transition from geometric thinking to algebraic thinking in the construction of third-degree remarkable identities, with particular attention to $(a + b)^3$. The instructional design relied on the analysis of a cube of side length $a + b$, inviting students to decompose the total volume into sub-volumes corresponding to algebraic monomials. Students were encouraged to

observe, segment, and interpret the three-dimensional configuration before producing the corresponding symbolic expression.

The written productions collected at the end of the sequence constituted the corpus for qualitative analysis. These data made it possible to identify the representational treatments and conversions performed by students, as well as the epistemic and semiotic obstacles that emerged during the coordination of geometric and symbolic registers. Particular attention was paid to the ways in which students articulated spatial structuring with algebraic expansion, thereby revealing the processes underlying the conceptualization of the identity.

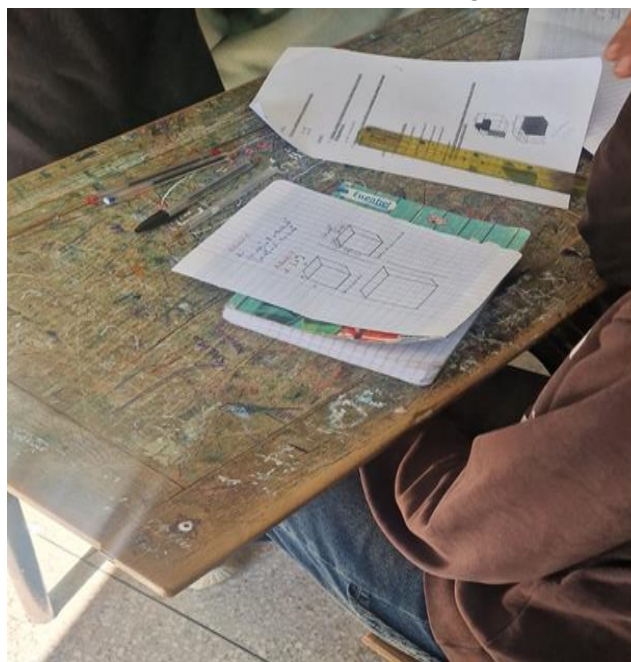


Figure 2 Students engaged in the volumetric modeling activity

4.1.

he Theoretical Phase of the Activity

The objective of the experiment was to provide a visual demonstration that the expansion of the algebraic expression $(a + b)^3$ corresponds to the sum of the volumes of several distinct solids. More specifically, the study investigates the relationship between spatial geometry and the remarkable identity $(a + b)^3$, with the aim of supporting the transition from spatial under-

standing (volume) to abstract understanding (algebraic computation).

The learning of third-degree remarkable identities is often limited to symbolic manipulation. This experimentation seeks to validate the algebraic expression $a^3 + 3a^2b + 3ab^2 + b^3$ through an empirical approach grounded in the manipulation of three-dimensional objects.

5. Materials and Method

Instructional support:

The instructional support consisted of an activity worksheet including representations in cavalier perspective. The worksheet was

Activités

Activité 1:
Développer algébriquement les expressions suivantes :

- $(a+b)^3 = \dots\dots\dots$
- $(a-b)^3 = \dots\dots\dots$

Activité 2:
1- Découper sous forme de volumes chacun des termes du développement suivantes :

$a^3 + 3a^2b + 3ab^2 + b^3$

2- Indiquer combien de volumes différents est nécessaires pour représenter l'expression suivantes :

$a^3 + 3a^2b + 3ab^2 + b^3$: volumes dessinées.

Activité 3:
1- Réaliser des dessins à la main de :

- Un cube d'arête a pour représenter a^3 .
- Trois volumes représentant les termes a^2b .
- Trois volumes représentant les termes ab^2 .
- Un cube d'arête b pour représenter b^3 .

a^3	a^2b	ab^2	b^3

Tools: The tools mobilized during the activity included:

- Freehand drawing in cavalier perspective,
- Algebraic computation,
- Physical assembly and visual construction of three-dimensional models.

4.1. methodology

The instructional sequence followed a structured progression:

Algebraic Expansion

Students first computed the expression $(a+b)^3$ in order to obtain $a^3 + 3a^2b + 3ab^2 + b^3$.

This step aimed to activate prior knowledge of remarkable identities and polynomial expansion.

Identification of Volumes

structured around progressive tasks designed to guide students from symbolic manipulation to geometric modeling and spatial interpretation.

2- Assemblage des volumes
i) Quelle forme géométrique obtient-on si on juxtapose l'une en dessous de autres deux formes géométriques de volumes abc et abd ?

Forme 1 : a^3 et a^2b	Forme 2 : a^2b et ab^2
Forme 3 : a^2b et ab^2	Forme 4 : ab^2 et b^3

ii) Déduire le volume totale obtenu à partir de l'assemblage :

Volume de $(a+b)^3$

3. Retrouver le même volume total à l'aide d'un raisonnement géométrique, sans avoir recours aux dessins précédents:

$(a+b)^3 = \dots\dots\dots$

$\dots\dots\dots$

$\dots\dots\dots$

$\dots\dots\dots$

$\dots\dots\dots$

Students were then invited to interpret each term of the expansion as a distinct volumetric component (cubes and rectangular prisms). The algebraic expression was decomposed into four categories of volumes corresponding to:

- one cube of volume a^3 ,
- one cube of volume b^3 ,
- M three rectangular prisms of volume a^2b ,
- three rectangular prisms of volume ab^2 .

Graphical Representation

Each algebraic term was represented as a three-dimensional solid in cavalier perspective, with explicit specification of edge lengths a and b . This stage required students to coordinate symbolic representation with spatial structuring.

Material Construction

Finally, students assembled cardboard models in order to physically reconstruct the cube of side length $a + b$, thereby validating the volumetric decomposition obtained symbolically.

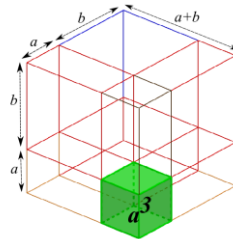
4.2.

Implementation of the Activity

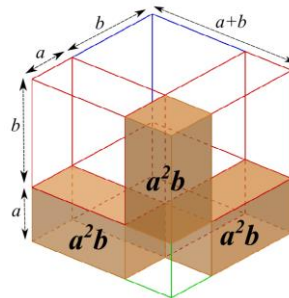
Step 1: Identification of the Eight Elementary Solids

Students first identified the eight elementary solids required for the volumetric decomposition of the cube of side length $a + b$.

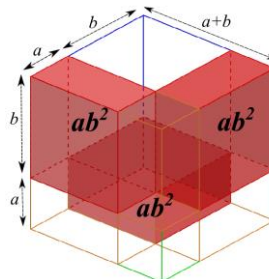
One cube with side length a , corresponding to a volume of a^3 .



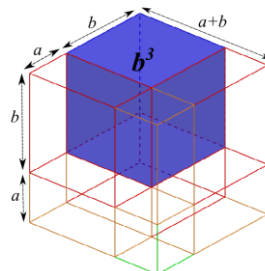
Three rectangular prisms with base area a^2 and height b , corresponding to a total volume of $3a^2b$.



Three rectangular prisms with base area b^2 and height a , corresponding to a total volume of $3ab^2$.



One cube with side length b , corresponding to a volume of b^3 .



This decomposition makes explicit the geometric structure underlying the algebraic expansion.

Step 2: Spatial Assembly

Students mentally or graphically assembled these eight solids to reconstruct a larger cube with total edge length $a + b$. This stage required the coordination of spatial reasoning and symbolic anticipation.

Step 3: Algebraic Verification

Students verified through calculation that the sum of the individual volumes equals the total volume of the large cube: $a^3 + 3a^2b + 3ab^2 + b^3 =$

This step consolidated the correspondence between geometric decomposition and algebraic expansion.

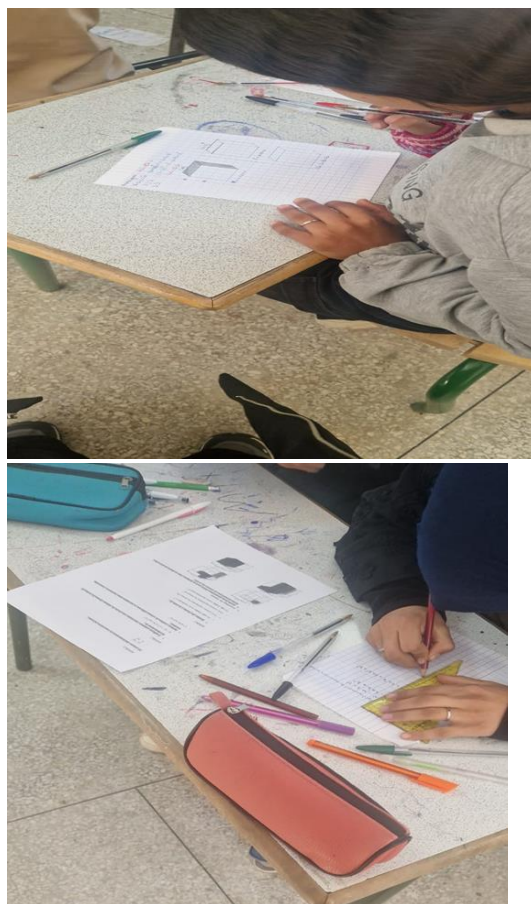


Figure 3 Students working on the theoretical worksheet

Summary / Observations:

The experiment demonstrates that the expression $(a + b)^3$ is not merely a formal algebraic rule, but represents the total volume

•

occupied by eight juxtaposed solids. In this way, geometry provides a concrete validation of algebraic expansion and supports the conceptualization of third-degree remarkable identities.

4.3.

the Practical Phase of the Activity

This experiment forms part of an approach integrating the manipulation of cardboard cubes as a semiotic mediation tool, aimed at promoting the spatial visualization of the identity $(a + b)^3$. The main challenge lies in students' difficulty conceptualizing third-degree powers beyond mere procedural computation. The manipulation of tangible materials (cardboard solids) facilitates the conversion from the symbolic register to the figural register, thereby reinforcing understanding of the internal structure of the formula.

The practical phase aimed to validate the remarkable identity

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

through the physical assembly of volumetric components.

Preparation of Materials

Construction of the eight solids:

- One cube with side length a , representing the monomial a^3 ;
- One cube with side length b , representing the monomial b^3 ;
- Three rectangular prisms with dimensions $a \times a \times b$, representing the term $3a^2b$;
- Three rectangular prisms with dimensions $a \times b \times b$, representing the term $3ab^2$.



Figure 4 Construction of the eight solids representing the terms of the expansion of $(a + b)^3$.

Manipulation Phase

Sorting step: Students identified and grouped the volumes according to their geometric structure, corresponding to the terms of algebraic expansion.

Assembly step: Students attempted to combine the eight solids to reconstruct a single cube with side length $a + b$.

Key observation: It was observed that removing any one of the pieces, for instance, one of the prisms corresponding to a^2b —prevented completion of the larger cube. This physically demonstrates that each algebraic term is essential for constructing the total volume.

6. Results and Interpretation

Anticipation phase: Students performed the classical algebraic expansion of $(a + b)^3$ and predicted the number and types of solids required.

Action phase: Students physically assembled the eight solids to reconstruct the cube of side length $a + b$. This step required managing spatial constraints and ensuring congruence of faces of lengths a and b .

Validation phase: Students compared the total volume obtained through physical assembly with the symbolic result of the polynomial expansion.

6.1. The analysis of the students' errors and difficulties.

The experiment highlights a strong structural correspondence between the algebraic formulation of the identity and its geometric realization. The manipulation of cardboard volumes as didactic mediators transforms an abstract identity into a tangible spatial object, thereby facilitating the understanding of the distribution and multiplicity of its terms. In this sense, the third-degree remarkable identity may be interpreted as the geometric decomposition of a segmented cubic volume.

Students no longer merely memorize a formula; rather, they visualize and construct a structured object.

However, despite this mediation, several categories of errors and difficulties were identified. These difficulties reveal important cognitive and semiotic tensions in the transition from geometric reasoning to algebraic generalization.

The main categories of errors identified during the experimentation, along with their possible didactic and cognitive origins, are summarized in Table 1.

Table 1 Classification of students' errors observed during the volumetric modeling activity, highlighting the interplay between symbolic manipulation, spatial reasoning, and semiotic coordination.

Error Category	Description of the Error	Didactic / Cognitive Origin
Algebraic errors (symbolic level)	- Incorrect expansion of powers - Omission or mismanagement of coefficients	- Overgeneralized use of distributivity - Lack of understanding of the coefficient as the number of identical volumes - Mechanical procedures without combinatorial or geometric meaning
Geometric errors (volume representation)	- Confusion between area and volume - Failure to distinguish volumes of the same type in the expression	- Difficulty in shifting between registers (Duval): algebraic ↔ 3D geometric - Coefficient perceived as decorative rather than indicating multiplicity
Geometric assembly errors	Inability to reconstruct the $(a + b)^3$ cube	Problem in spatial structuring
Mathematical reasoning errors	Inability to relate geometry and algebra	- Algebra disconnected from geometry - Procedural learning without conceptual understanding

These categories of errors do not represent isolated misconceptions but reflect tensions inherent in the coordination of semiotic registers and the progressive abstraction of spatial structures.

a. Algebraic Errors (Symbolic Level)

Error 1: Incorrect expansion of powers

Several students incorrectly expanded the expression by overgeneralizing distributive reasoning. For example, some wrote: $(a + b)^3 = 3a + 3b$ instead of performing the full polynomial expansion.

This type of response reflects a confusion between exponentiation and multiplication, as well as an overextension of linear distributive rules to higher-degree expressions.

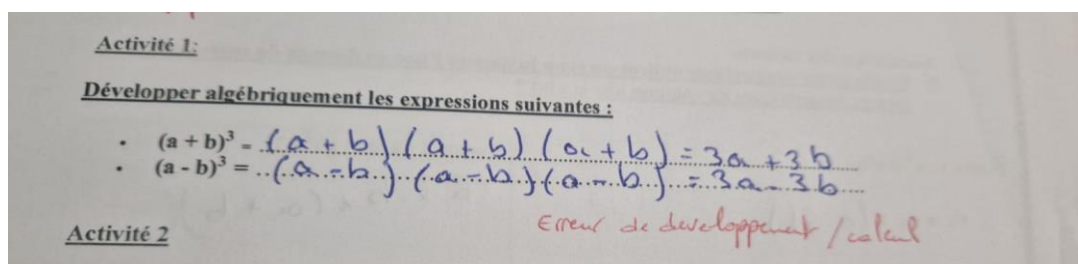


Figure 5 Example of incorrect expansion of $(a + b)^3$ reduced to a linear expression, illustrating overgeneralization of distributive reasoning.

Other students attempted expansion but failed to structure the polynomial correctly, as shown below:

$$(a + b)^3 = a^3 + 3ab + b^3$$

In this case, intermediate terms were either incomplete or improperly constructed, indicating partial procedural knowledge without full structural understanding.

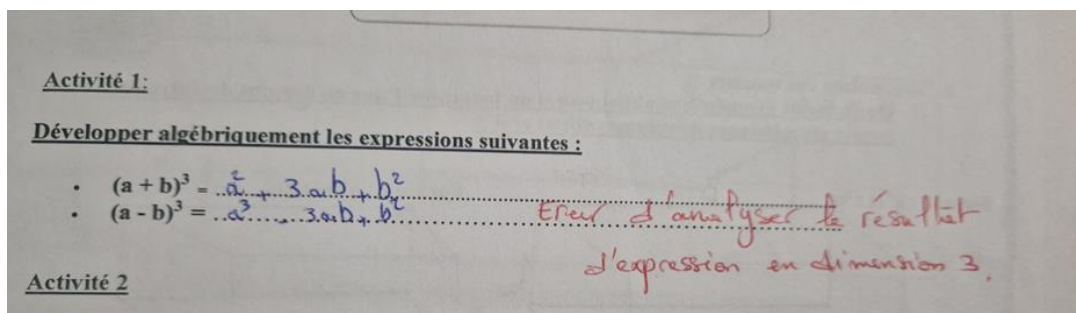


Figure 6 Student production showing incomplete cubic expansion, revealing partial procedural knowledge and misunderstanding of term multiplicity.

Origin of Error 1: Overgeneralized use of distributivity, resulting in incorrect extension of rules for powers.

Error 2: Omission or mismanagement of coefficients

Examples observed:

$$a^3 + ab^2 + a^2b + b^3$$

(coefficients 3 omitted)

$$a^3 + 3a^2b + ab^2 + b^3$$

(coefficients partially misassigned)

These productions indicate that students recognized the types of monomials involved but failed to account for their multiplicity.

Origin of Error 2

Lack of understanding of the coefficient as representing the number of identical volumes

Mechanical procedure without combinatorial or geometric meaning

These algebraic errors suggest that students tend to operate at a procedural level, without coordinating symbolic manipulation with the underlying combinatorial and geometric structure of the expression.

b. Geometric Errors (Volume Representation)

Error 3: Confusion between area and volume
Students draw rectangles instead of rectangular prisms, representing expressions as surfaces rather than volumes.

This error reveals a conceptual confusion between area (2D measure) and volume (3D measure), as well as difficulty in visualizing and representing the third dimension.

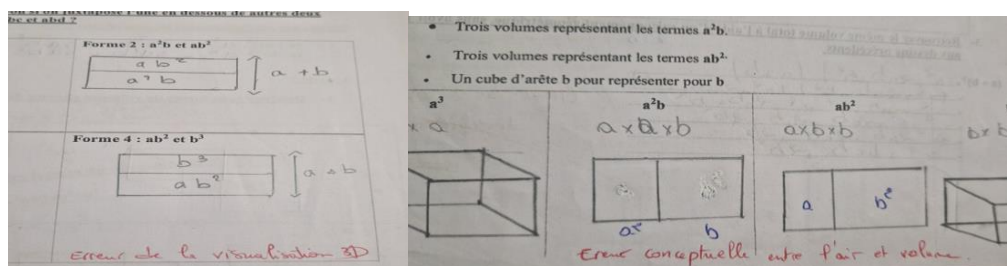


Figure 7 Student productions illustrating the confusion between two-dimensional surface representations and three-dimensional volumetric decomposition.

Origin of Error:

Difficulty shifting between semiotic registers (Duval, 1995, 2006): algebraic $\leftrightarrow$ three-dimensional geometric; insufficient mastery of spatial visualization.

Error 4: Failure to distinguish volumes of the same type
Students draw only a single solid to represent $3a^2b$ or $3ab^2$ or fail to recognize that these correspond to three identical volumes.

This difficulty reflects an incomplete coordination between symbolic coefficients and geometric repetition.

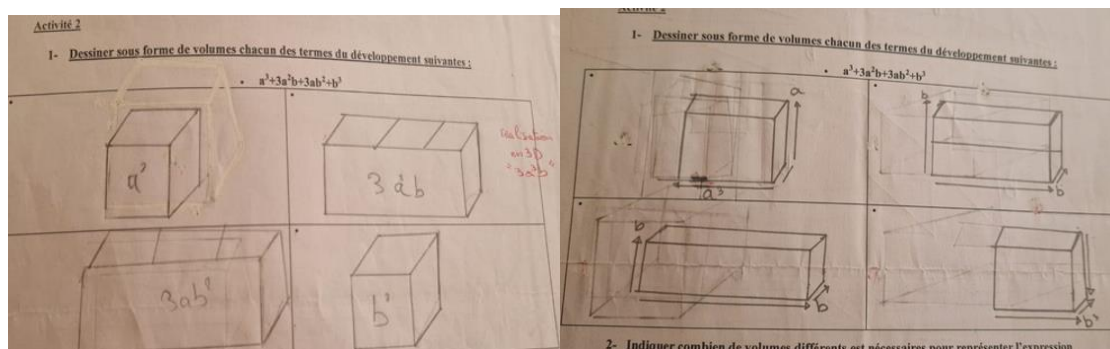


Figure 8 Student productions illustrating the failure to represent the multiplicity of identical volumes in the expansion of $(a + b)^3$.

Origin of Error :
Students treat the coefficient not as an indication of multiplicity.

c. Geometric Assembly Errors

Error 5: Inability to Reconstruct the Cube $(a + b)^3$

During assembly, the volumes are juxtaposed without coherence, preventing the student from forming a complete cube.

This error reflects difficulties in coordinating spatial constraints and maintaining a holistic representation of the cubic configuration.

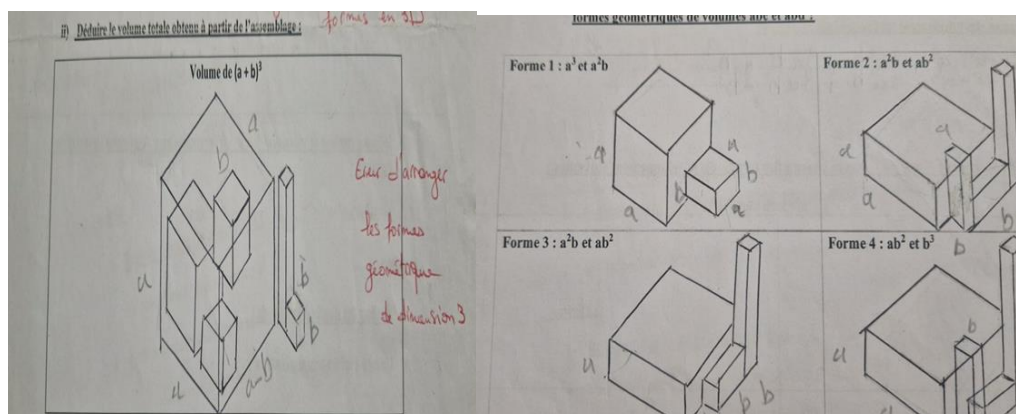


Figure 9 Student attempts to assemble the volumetric components, illustrating difficulties in reconstructing the cube of edge $(a + b)$.

Origin of Error 5:

Localized perception of objects (volume by volume reasoning), lack of a global view of the cubic structure, and difficulties in spatial structuring.

d. **Mathematical Reasoning Errors**

Error 6: Inability to connect geometry and algebra

Even after assembling the cube, some students fail to recover the corresponding algebraic expression or to relate it to $(a + b)^3$.

This indicates that the embodied experience did not automatically lead to algebraic abstraction.

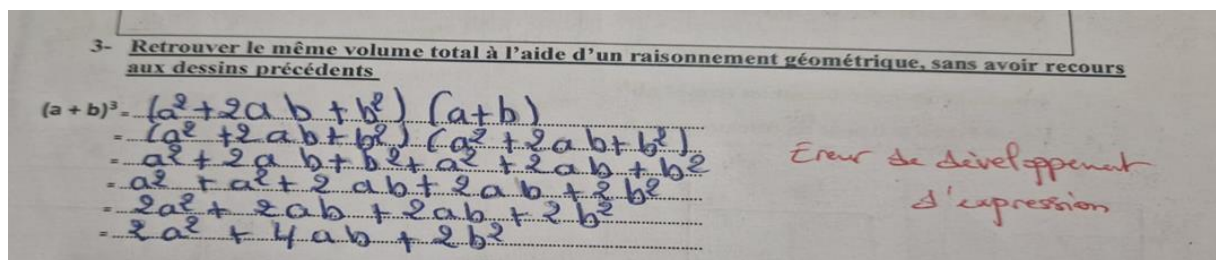


Figure 10 Illustration of the failure to translate embodied geometric experience into coherent algebraic abstraction.

Origin of Error 6: Algebra disconnected from geometry; procedural learning without conceptual understanding.

6.2. Analysis of Results

a. Results for Third-Year Middle School Students

In this section, we present the number and percentage of students according to their responses for each activity (theoretical/practical).

The results are presented quantitatively, using tables and graphs.

Table 2 presents the percentage of correct responses obtained in the theoretical and practical phases for second-year middle school students.

Table 2 Percentage of correct responses in the theoretical and practical phases for second-year middle school students.

Second Year of Middle School				
Class	2/3 (44 students)		2/4 (42 students)	
Activity	Theoretical	Practical	Theoretical	Practical
Volume Representation	13 (29,5%)	36(81,9 %)	15(35,41%)	32(76,2%)
Geometric Assembly	21(47,7%)	40(90,9%)	11(26,2%)	39(92,9%)
Deduction of $(a+b)^3$	11(25%)	42(95,45%)	10(23,9%)	36(85,8%)

To facilitate the interpretation of the table data, the results are presented below in the form of bar charts. Figures 11 visually illustrate the comparison between the theoretical and practical phases for second-year middle school students, highlighting the performance gains observed after the manipulation activity.

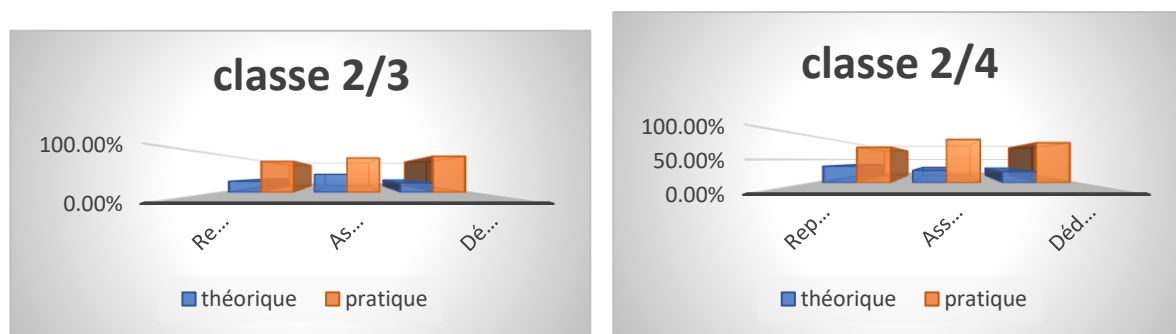


Figure 11 Comparison of theoretical and practical performance : Class 2/3 and 2/4

The data show a substantial improvement across all tasks following the practical phase. The most significant gains are observed in the deduction of the cubic identity, where success rates increased by more than 60 percentage points in both classes. This suggests that symbolic generalization alone was insufficient, whereas volumetric manipulation acted as a mediating support for structural understanding.

b. Results for Third-Year Middle School Students

Table 3 presents the percentage of correct responses obtained in the theoretical and practical phases for third-year middle school students (Classes 3/1 and 3/2).

Table 3 Percentage of correct responses in the theoretical and practical phases for third-year middle school students.

Third Year of Middle School				
Class	3/1 (44 students)		3/2 (43 students)	
Activity	Theoretical	Practical	Theoretical	Practical
Volume Representation	24 (54,6%)	40(91 %)	19(44,18%)	39(90,6%)
Geometric Assembly	26(59%)	42(95,45%)	18(41,85%)	40(93%)
Deduction of $(a+b)^3$ Expansion	23(52,2%)	42(95,45%)	17(39,5%)	42(97,68%)

To facilitate the interpretation of the table data, the results are presented below in the form of bar charts.

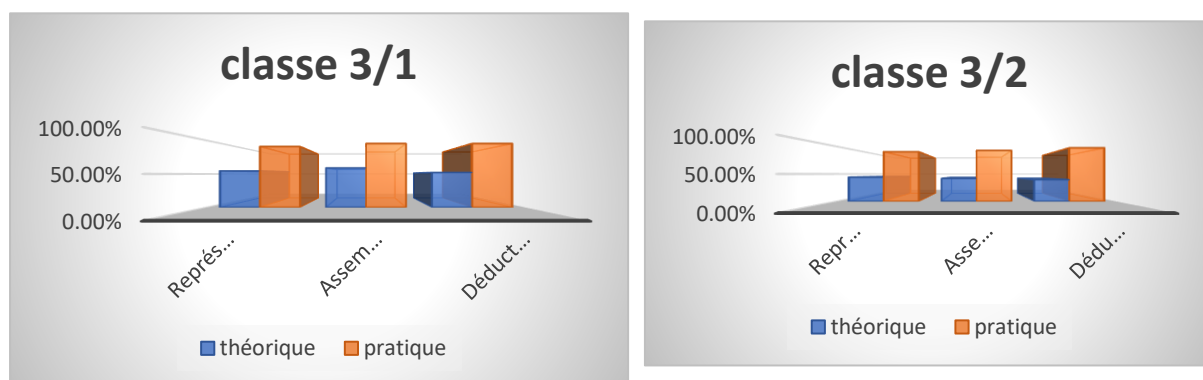


Figure 12 Comparison of theoretical and practical performance – Class 3/1 and 3/2.

The results show a substantial improvement across all three tasks following the practical phase. Although third-year students demonstrated stronger initial performance during the theoretical phase compared to second-year students, the manipulation activity led to near-universal mastery in both classes. The most pronounced gains are observed in the deduction of the cubic identity, where performance increased by more than 40 percentage points.

These findings suggest that prior symbolic familiarity does not eliminate structural misunderstandings. Rather, the volumetric manipulation appears to consolidate the coordination between geometric and algebraic registers, strengthening students' structural comprehension of the identity.

From a global didactic perspective, the theoretical phase highlights persistent difficulties that can be interpreted through established theoretical frameworks. First, students encounter obstacles in shifting between semiotic registers (Duval, 1995), particularly in coordinating the geometric register (3D representation) with the symbolic algebraic register. Second, the coefficient is often not conceptualized as an indicator of multiplicity but treated procedurally, revealing a fragile structural understanding. Third, the transition from geometric to algebraic reasoning appears incomplete, reflecting a gap between perceptual and analytical modes of thought.

According to van Hiele's model, many students operate at levels characterized by visualization or informal analysis, without reaching a level where structural relationships are explicitly coordinated. The manipulation of cardboard cubes functions as a mediating device that supports progression toward a more relational and deductive level of reasoning. In Radford's (2018) terms, the activity promotes algebraic thinking by enabling students to treat indeterminate quantities analytically while grounding symbolic expressions in culturally meaningful spatial actions.

The embodied reconstruction of the cube of edge length $(a + b)$ makes visible the combinatorial necessity of the six prisms and two cubes. This material coordination between action and symbol fosters structural awareness and strengthens the link between spatial reasoning and algebraic generalization.

7. Discussion

The results suggest that the transition from geometric thinking to algebraic thinking is neither spontaneous nor guaranteed by symbolic instruction alone. The theoretical phase reveals significant difficulties in coordinating semiotic registers, confirming Duval's (1995) claim that conceptual understanding depends on the ability to convert and coordinate representations. Students often manipulate symbols procedurally without integrating their geometric meaning.

Moreover, the weak conceptualization of coefficients as multiplicities indicates a fragile structural view of algebraic expressions. In many cases, equality is treated operationally rather than relationally, limiting students' capacity to perceive the identity as a structured whole. The practical manipulation phase appears to function as a semiotic mediator. In van Hiele's terms, the activity supports movement from visualization toward relational reasoning by confronting students with the structural necessity of each volumetric component. The impossibility of completing the cube when one prism is removed materializes the combinatorial logic of the identity.

In line with Radford's (2018) conception of algebraic thinking, the activity promotes the analytical treatment of indeterminate quantities through culturally meaningful actions. The physical reconstruction of the cube of edge length $(a + b)$ transforms the identity from a memorized formula into a structured spatial organization. The embodied coordination between action, perception, and symbol fosters deeper structural awareness and supports the generalization process. These findings suggest that volumetric modeling can serve as an effective didactic device for

supporting the genesis of algebraic understanding, particularly in the context of higher-degree polynomial identities.

Conclusion.

This study made it possible to identify and analyze the difficulties encountered by middle school students in learning three-dimensional remarkable identities, particularly during the transition from geometric thinking to algebraic reasoning. The analysis of students' productions shows that the observed errors cannot be reduced to simple computational mistakes; rather, they reflect deeper didactic, conceptual, and semiotic obstacles related to the coordination of representations and the construction of structural meaning.

Translating these concerns into effective classroom practices remains a significant challenge. One of the central aims of this research was therefore to identify instructional conditions capable of supporting the development of algebraic thinking. As emphasized in previous research (Labelle, 2008), understanding how students progressively construct algebraic knowledge is essential for designing meaningful didactic interventions.

The results underline that the generalization of three-dimensional remarkable identities, limited combinatorial reasoning, and weak coordination between geometric and symbolic registers constitute major barriers to constructing a meaningful understanding of the identity

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

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- Many students struggled to interpret coefficients as indicators of multiplicity and to perceive the structural organization underlying the expansion.
- The data collected confirms the didactic relevance of physical representations through cardboard cubes. These material supports act as mediators between the perceptual and the symbolic, enabling students to progressively move from concrete manipulation to abstract formulation. The embodied reconstruction of the cubic structure appears to foster spatial reasoning, strengthen register coordination, and promote a more structural view of algebraic expressions.
- From a didactic perspective, this study highlights the importance of designing learning situations that integrate hands-on manipulation, spatial visualization, and explicit coordination between semiotic registers. Such approaches allow students to move beyond purely formal memorization of remarkable identities and instead develop durable conceptual understanding grounded in structural reasoning.
- Finally, this research opens several avenues for further investigation. Future studies may explore the integration of three-dimensional digital visualization tools, examine the long-term retention effects of embodied modeling, and analyze the progressive curricular development of remarkable identities across grade levels. Such work may contribute to improving both instructional practices and students' appropriation of algebraic structure.

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